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# The Infinite Curvature Limit of AdS/CFT[1]

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Some kinematical speculations on the infinite curvature limit of the conjectured duality of Maldacena between 10-dimensional strings living in $AdS_5 \times S_5$ and an ordinary 4-dimensional quantum field theory, namely $\mathcal{N} = 4$ super Yang–Mills with gauge group $SU(N)$, are given.

**KEY WORDS:** infinite curvature limit.

## 1. INTRODUCTION

The usual AdS/CFT correspondence relates strings living on a manifold of curvature

$$R \sim \frac{1}{l^2} \tag{1}$$

to an ordinary four-dimensional conformal field theory (CFT) with gauge coupling $g$ given in terms of the string coupling constant, $g_s$, by

$$g = g_s^{1/2} \tag{2}$$

The 't Hooft coupling is

$$\lambda \equiv g^2 N \equiv g_s N \tag{3}$$

Both the 't Hooft coupling and the effective string tension are given in terms of the string length $l_s^2 \equiv \alpha'$ by

$$\lambda^{1/2} = \frac{l^2}{l_s^2} \sim T_{\text{eff}} \tag{4}$$

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and the 10-dimensional Newton constant is

$$\kappa_{10}^2 \sim l_p^8 = g_s^2 l_s^8 \sim \frac{l^8}{N^2} \tag{5}$$

The correspondence is usually studied in the low curvature regime in which

$$\frac{l}{l_s} \gg 1 \tag{6}$$

In this regime strings are believed to be well approximated by supergravity, but CFT is strongly coupled. Many nontrivial checks are however possible owing to the existence of gauge invariant operators protected by supersymmetry whose correlators are total or partially determined through kinematics. On the string side these correlators are determined by computing the action for the relevant fields in terms of arbitrary sources at the conformal boundary.

The opposite limit, i.e.,

$$\frac{l}{l_s} \ll 1 \tag{7}$$

corresponds to effectively tensionless strings in a strongly curved (and, as we shall see, somewhat singular) background. The corresponding CFT is, however, in the perturbative regime.

On the string theory side, however, it is not even clear what is the meaning of the sources, and it is not known how to decode the information suposedly provided by the preturbative CFT.

The purpose of the present note is the quite modest one of discussing the high curvature limit of *AdS*, and to argue that it is none other than the light cone itself.

## 2. THE INFINITE CURVATURE LIMIT OF CONSTANT CURVATURE SPACES

Constant curvature spaces of any signature can be understood (cf. for example, Alvarez, 2003) as hypersurfaces of flat $n$-dimensional space with metric

$$ds^2 = \sum_{a=1}^{n} \epsilon_a \, dx_a^2 \tag{8}$$

where all $\epsilon_a = \pm 1$. The signs are arbitrary, *except* for the condition that at least one coordinate, but not all of them, has got to be time-like, which in our conventions means positive sign.

Calling $x_{n-1}$ one of the time-like coordinates, and $x_n$ one of the space-like ones, this means that the metric enjoys the term

$$dx_{n-1}^2 - dx_n^2 \tag{9}$$

The equation determining the surface itself is

$$\sum_{a=1}^{n} \epsilon_a x_a^2 = \pm l^2 \tag{10}$$

Here the length scale $\ell$ determined the curvature through

$$R = \pm \frac{n(n+1)}{l^2} \tag{11}$$

All these manifolds enjoy a maximal group of isometries, which is a real form of $SO(n)$. The Killings are given by

$$L_{ab} \equiv \epsilon_a x^a \partial_b - \epsilon_b x^b \partial_a \tag{12}$$

(no Einstein implicit sum convention is applied in this definition). Horospheric coordinates are defined by

$$\begin{aligned} x_- &\equiv x_n - x_{n-1} \\ z &\equiv \frac{l}{x_-} \\ y^i &\equiv z x^i i = 1 \ldots n-2 \end{aligned} \tag{13}$$

The metric reads in general

$$ds^2 = \frac{\sum_i \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{14}$$

The case corresponding to our present interest is when

$$\epsilon_i = -1 \forall i \tag{15}$$

It has isometry group $SO(1, n)$ and metric

$$ds^2 = \frac{-\sum_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{16}$$

The lorentzian form is de Sitter space, and the euclidean form is what is usually called *euclidean Anti de Sitter*, although it could equally well be called *euclidean de sitter*. (There are no euclidean versions with isometry group $SO(2, n)$.)

Written in this form, it is quite obvious that when $l \to 0$, which corresponds to the equation

$$x_{n-1}^2 - \sum_{i=1}^{n-2} x_i^2 - x_n^2 = 0 \tag{17}$$

is the light cone of the origin in ordinary $n$-dimensional Minkowski space, which we will denote by $N_{\pm}(0)$, with metric

$$ds^2 = \frac{-\sum_i dy_i^2}{z^2} \tag{18}$$

## 3. LIFE ON THE LIGHT CONE

The local structure of the light cone is $S^{n-2} \times \mathbb{R}^+$, and a point in $N_+$ can be specified by $(x_0, n^i)$, where $x_0 \in \mathbb{R}^+$ and $\vec{n}^2 = 1$ is a point on the unit $(n-2)$-dimensional sphere, $S_{n-2}$, that is, an $(n-1)$-dimensional structure. The light cone can be visualized as an $S_{n-2}$ sphere of radius $x_0$.

The induced metric $h_{ij}$ is, however, degenerate (i.e., as a matrix it has rank $n$), because the time differential is totally absent from the line element:

$$ds_+^2 = x_0^2\, d\Omega_{n-2}^2 \tag{19}$$

where $d\Omega_n^2$ is the metric on the unit $n$-sphere, $S_n$, which in terms of angular variables, reads

$$d\Omega_n^2 \equiv d\theta_n^2 - \sin\theta_n^2\, d\theta_{n-1}^2 + \cdots + \sin\theta_n^2 \sin\theta_{n-1}^2 \ldots \sin\theta_2^2\, d\theta_1^2 \tag{20}$$

This means that, although singular as a metric on $N_+$, the metric is perfectly regular (actually the standard one) as a metric on the $(n-2)$-spheres $t =$ constant.

The invariant volume element, however, vanishes because of the fact that

$$\sqrt{h} = 0 \tag{21}$$

Remarkably enough, and in spite of some statements on the contrary, the complete set of isometries of the *three*-dimensional metric (20) includes the full four-dimensional Lorentz group, $SO(1, 3)$.[4] This should be quite obvious from our limiting process of the previous paragraph.

The six Killing vectors with factorizable coeficients which generate $SO(1, 3)$ are actually given by

$$J_1 = \cos\phi \frac{\partial}{\partial\theta} - \cot\theta\, \sin\phi \frac{\partial}{\partial\phi} \tag{22}$$

What it is perhaps not immediatly obvious is that this is not the full history; it will be shown in the next section that there is actually an infinite dimensional group of isometries.

Also interesting are those transformations that leave invariant the metric up to a Weyl rescaling. Those are the conformal isometries which in four dimensions

[4] Isometries are well-defined, even for singular metrics, through the vanishing Lie-derivative condition $\pounds(k) g_{\mu\nu} = 0$, reflecting the invariance of the metric under the corresponding one-parametric group of diffeomorphisms, although of course this is not equivalent to $\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$ because the covariant derivative (i.e., the Christoffel symbols) is not well defined owing to the absence of the inverse metric.

span the group called by Penrose and Rindler the Newman–Unti (NU) group (cf. Penrose and Rindler, 1984) i.e.,

$$\begin{aligned} x^0 &\to F(x^0, z, \bar{z}) \\ z &\to \frac{az+b}{cz+d} \end{aligned} \tag{23}$$

The NU group is an infinite dimensional extension of the Möbius group.

## 4. DEGENERATE HOROSPHERIC COORDINATES

It could appear curious that when writing the metric of the cone $N_+$ in terms of the degenerate horospheres as in Eq. (18) translation invariance is apparent in the coordinates $(y_1, y_2)$. Physically what happens is that those coordinates are a sort of stereographic projection, singular when $x^0 = x_3$. The exact relationship between cartesian and horospheric coordinates in the infinite curvature limit is

$$\begin{aligned} x_0 &= \frac{1}{2z}\left(y_T^2 + 1\right) \\ x_3 &= \frac{1}{2z}\left(y_T^2 - 1\right) \\ x_{\mathrm{T}} &= \frac{y_T}{z} \end{aligned} \tag{24}$$

where the subscript *transverse* refers to the (1, 2) labels: $y_T \equiv (y_1, y_2)$. It is worth pointing out that the coordinate z has got dimensions of energy whereas the $y_T$ are dimensionless.

Horospheric coordinates then break down when $x_0 = x_3$; that is, when $z = \propto$. It is a simple matter to recover the Killings corresponding to the Lorentz subgroup:

$$\begin{aligned} J_1 &= -zy_1\frac{\partial}{\partial z} - \frac{1}{2}(y_1^2 - y_2^2 - 1)\frac{\partial}{\partial y_1} + y_1y_2\frac{\partial}{\partial y_2} \\ J_2 &= -zy_2\frac{\partial}{\partial z} - y_1y_2\frac{\partial}{\partial y_1} + \frac{y_1^2 - y_2^2 - 1}{2}\frac{\partial}{\partial y_2} \\ J_3 &= y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} \\ K_1 &= y_2z\frac{\partial}{\partial z} - y_1y_2\frac{\partial}{\partial y_1} + \frac{y_2^2 - y_1^2 - 1}{2}\frac{\partial}{\partial y_2} \\ K_2 &= -y_1z\frac{\partial}{\partial z} - \frac{y_2^2 + 1 - y_1^2}{2}\frac{\partial}{\partial y_1} - y_1y_2\frac{\partial}{\partial y_2} \\ K_3 &= z\frac{\partial}{\partial z} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} \end{aligned} \tag{25}$$

But there are more Killing vectors. First of all, the two translational ones. obvious in these coordinates,

$$P_1 \equiv \frac{\partial}{\partial y_1}$$
$$P_2 \equiv \frac{\partial}{\partial y_2} \tag{26}$$

and some others, such as

$$\begin{aligned}
L_1 &= e^{y_1}\left(\cos y_2\left(z\frac{\partial}{\partial z}+\frac{\partial}{\partial y_1}\right)+\sin y_2\frac{\partial}{\partial y_2}\right)\\
L_2 &= e^{y_1}\left(\sin y_2\left(z\frac{\partial}{\partial z}+\frac{\partial}{\partial y_1}\right)-\cos y_2\frac{\partial}{\partial y_2}\right)\\
J_3 &= e^{y_2}\left(\cos y_1\left(z\frac{\partial}{\partial z}+\frac{\partial}{\partial y_2}\right)+\sin y_1\frac{\partial}{\partial y_1}\right)\\
L_4 &= e^{y_2}\left(\sin y_1\left(z\frac{\partial}{\partial z}+\frac{\partial}{\partial y_2}\right)-\cos y_1\frac{\partial}{\partial y_1}\right)\\
L_5 &= e^{-y_1}\left(\cos y_2\left(z\frac{\partial}{\partial z}-\frac{\partial}{\partial y_1}\right)+\sin y_2\frac{\partial}{\partial y_2}\right)\\
L_6 &= e^{-y_1}\left(\sin y_2\left(z\frac{\partial}{\partial z}-\frac{\partial}{\partial y_1}\right)-\cos y_2\frac{\partial}{\partial y_2}\right)\\
L_7 &= e^{-y_2}\left(\cos y_1\left(z\frac{\partial}{\partial z}-\frac{\partial}{\partial y_2}\right)+\sin y_1\frac{\partial}{\partial y_1}\right)\\
L_8 &= e^{-y_2}\left(\sin y_1\left(z\frac{\partial}{\partial z}-\frac{\partial}{\partial y_2}\right)-\cos y_1\frac{\partial}{\partial y_1}\right)
\end{aligned} \tag{27}$$

More Killings are gotten through commutation; the boost $K_3$, in particular, raises powers of the coordinates when acting on $L$s:

$$\begin{aligned}
[K_3, L_1] &= e^{y_1}\Bigg((y_1\ \cos y_2 - y_2\ \cos y_2)z\frac{\partial}{\partial z} + ((y_1-1)\ \cos y_2 - y_2 \cos y_2)\\
&\quad \times\frac{\partial}{\partial y_1} + ((y_1-1)\ \cos y_2 + y_2\ \cos y_2)\frac{\partial}{\partial y_2}\Bigg) \equiv Q_1\\
[K_3, L_2] &= e^{y_1}\Bigg((y_1\ \cos y_2 - y_2\ \cos y_2)z\frac{\partial}{\partial z} + (-(y_1-1)\ \cos y_2 + y_2 \cos y_2)\\
&\quad \times\frac{\partial}{\partial y_2} + ((y_1-1)\ \cos y_2 + y_2\ \cos y_2)\frac{\partial}{\partial y_1}\Bigg) \equiv Q_2
\end{aligned} \tag{28}$$

Clearly the process never ends. Commuting again with $K_3$ produces terms in $y_1^2 e^{y_1}$ which are not found amongst the existing generators. The isometry group is then infinite dimensional.

It is actually possible to give the general solution of the Killing equation in closed form using horospheric coordinates. Given an arbitrary analytic function of the complex variable $y_1 + iy_2$, for example $f(y_1 + iy_2)$, it is given by

$$k \equiv \left(\frac{\partial^2}{\partial y_1^2} Ref\right) z\frac{\partial}{\partial z} + \left(\frac{\partial}{\partial y_1} Ref\right)\frac{\partial}{\partial y_1} - \left(\frac{\partial}{\partial y_2} Ref\right)\frac{\partial}{\partial y_2} \tag{29}$$

It is now clear that the isometry group of the four-dimensional light cone $N_+$ is an infinite dimensional group, which includes the Lorentz group as a subgroup.

We find this to be a remarkable situation.

## 5. CONCLUSION: SIGMA MODELS ON SINGULAR MANIFOLDS

We can expect this approximation to work for length scales much larger than the one defined by the curvature inverse, i.e., it is a low energy approximation, valid for $E \ll l^{-1}$.

The (singular) propagator boundary–boundary we get in this way (when $l \equiv \epsilon \to 0$) is

$$\Delta_{b-b} \equiv \frac{\epsilon^{n+1}\Gamma(n-1)}{\pi^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)} \frac{z^{n-1}}{|\vec{y} - \vec{y}'|^{n-2}} \tag{30}$$

It is quite difficult however to make progress along these lines in a sigma model approach. For example, the usual representation of $AdS_3$ as a Wess–Zumino–Witten (WZW) model leads to the lagrangian:

$$L = 2k\left(\frac{1}{u^2}\partial u\bar{\partial} u + u^2 \partial\bar{\gamma}\bar{\partial}\gamma\right) \tag{31}$$

(where $(u, \gamma, \bar{\gamma})$ are coordinates descibed in detail in Giveon and Kutasov, 2002). The parameter

$$k = l^2 \tag{32}$$

so that in the degenerate limit $k = 0$. But this is bad, because the central charge of the underlying CFT is

$$c = \frac{3k}{k-2} \tag{33}$$

so that usual considerations are restricted to k $>$ 2. More work on these issues can be, however, rewarding.

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